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Another Fourier-style expansion in series of Legendre functions

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Abstract

In Love (Proc. London Math. Soc. 69 (3) (1994) 629–672) I established, under suitable conditions, a Fourier-style expansion, in Legendre functions $\mathrm{P}_\nu^\mu(x)$ on the cut $-1<x<1$, of the form

$$\frac{1}{2}\{f(x+0)+f(x-0)\}=a_0\mathrm{P}_\nu^\mu(x)+\sum_{n=1}^{\infty}\{a_n\mathrm{P}_{\nu+n}^\mu(x)+a_{-n}\mathrm{P}_{\nu-n}^\mu(x)\}.$$

In this paper I establish an expansion of similar form but in which the terms corresponding to odd values of n are absent. This is achieved by adding to the above an expansion of zero of similar form; most of this paper is devoted to establishing that expansion of zero. Expansions similar to these occur in Love and Hunter (Proc. London Math. Soc. 64 (3) (1992) 579–601), but under conditions which are more restrictive; the methods used in this paper are mostly quite different.

Keywords: Legendre functions (not polynomials); Expansion of zero; Littlewood's Tauberian Theorem for Abel summability

1. Introduction

The Legendre functions in this paper are defined (as in [1, 3.4(6), p. 143]) by

$$\mathrm{P}_\nu^\mu(x)=\frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2}F\left(\begin{matrix}-\nu,1+\nu; & \dfrac{1-x}{2}\\ 1-\mu; & \end{matrix}\right),$$

where F is Gauss's hypergeometric function and μ and ν are real or complex parameters, $-1<x<1$, and the power has its principal value.

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In [3, Theorem 4] I established the following expansion theorem:
If $(1-t^2)^{-1/4}f(t) \in L(-1,1)$, f *has bounded variation on a neighbourhood of a certain* $x \in (-1,1)$, $|\mathrm{re}\,\mu| < \frac{1}{2}$, v *is not half an odd integer, and*

$$a_n = (-1)^n \frac{v+n+\frac{1}{2}}{2\cos v\pi} \int_{-1}^{1} f(t)\mathrm{P}_{v+n}^{-\mu}(-t)\,\mathrm{d}t,$$

then

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = a_0\mathrm{P}_v^\mu(x) + \sum_{n=1}^{\infty}\{a_n\mathrm{P}_{v+n}^\mu(x) + a_{-n}\mathrm{P}_{v-n}^\mu(x)\}.$$

In the present I aim to prove (in Theorem 3 herein), *with the additional condition that* f *has bounded variation on a neighbourhood of* $-x$, *and* $c_n = (-1)^n 2a_n$, *that*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = c_0\mathrm{P}_v^\mu(x) + \sum_{n=1}^{\infty}\{c_{2n}\mathrm{P}_{v+2n}^\mu(x) + c_{-2n}\mathrm{P}_{v-2n}^\mu(x)\}.$$

In these two expansions, the latter omits every second Legendre function occurring in the former. The latter is obtained by a linear combination of the former and an “expansion of zero”. Most of the present paper is devoted to establishing this expansion of zero (Theorem 2 herein). The first few lemmas are very similar to those in [3]; their proofs are drastically curtailed here, because in full they would be almost word for word the same as in [3]. But the subsequent work is quite different from that in [3].

Theorems 2 and 3 resemble Theorems 8 and 11 in [2], the difference being that bounded variation on a neighbourhood of x or $-x$ replaces the Dini condition. Neither of these conditions includes the other; but the bounded variation condition seems to be less restrictive in that it permits f to have ordinary discontinuities whereas the Dini condition does not.

2. Preliminaries

Throughout the paper μ and v are fixed complex numbers, $|\mathrm{re}\,\mu| < \frac{1}{2}$, and v is not half an odd integer; f is a complex-valued function such that $(1-t^2)^{-1/4}f(t) \in L(-1,1)$. Also $\omega = \arccos x$ and $\theta = \arccos t$, and ω is usually fixed in the later work. Further $0 < r < 1$. These hypotheses are understood, but not always stated, in the lemmas.

Lemma 1. *If* $0 < \alpha < \beta < \pi$, f *has support* $[\cos\beta, \cos\alpha]$,

$$b_n = \frac{v+n+\frac{1}{2}}{2\cos v\pi} \int_{-1}^{1} f(t)\mathrm{P}_{v+n}^{-\mu}(-t)\,\mathrm{d}t, \tag{1}$$

and

$$B(r) = \sum_{n=-\infty}^{\infty} r^{|n|} b_n \mathrm{P}_{v+n}^\mu(\cos\omega), \tag{2}$$

then

$$B(r) = \frac{\cos\mu\pi \sin^\mu\omega}{\pi^2\cos v\pi} I(r), \tag{3}$$

where

$$I(r) = \int_0^{\omega} \frac{\mathrm{d}\phi}{(\cos\phi - \cos\omega)^{(1/2)+\mu}} \int_{\alpha}^{\beta} g(\theta)\sin\theta\,\mathrm{d}\theta \int_{\theta}^{\pi} \frac{\mathrm{d}\psi}{(\cos\theta - \cos\psi)^{(1/2)-\mu}} S(r,\phi,\psi), \tag{4}$$

$$g(\theta) = f(\cos\theta)\sin^{-\mu}\theta, \tag{5}$$

$$S(r,\phi,\psi) = \sum_{n=-\infty}^{\infty} r^{|n|} v_n \cos v_n\phi \cos v_n(\pi - \psi), \tag{6}$$

and

$$v_n = v + n + \tfrac{1}{2}.$$

Proof. As in [3, Lemma 1] with r replaced by $-r$. As remarked there, the series S and the integrals are absolutely convergent, so that changes of order are justified. □

Note. The symbol v_n has the same meaning here as in [3], but this is different from its meaning in [2].

Lemma 2. *If* $0 < \alpha < \beta < \pi$, f *has bounded variation on* $[\cos\beta, \cos\alpha]$, $\kappa' = \frac{1}{2} + \mu$, *and* $J(r,\phi)$ *is the inner double integral in* (4), *namely*

$$J(r,\phi) = \int_{\alpha}^{\beta} g(\theta)\sin\theta\,\mathrm{d}\theta \int_{\theta}^{\pi} \frac{\mathrm{d}\psi}{(\cos\theta - \cos\psi)^{(1/2)-\mu}} S(r,\phi,\psi), \tag{7}$$

then

$$\begin{aligned}\kappa' J(r,\phi) = &- g(\beta)\int_{\beta}^{\pi} S(r,\phi,\psi)(\cos\beta - \cos\psi)^{\kappa'}\,\mathrm{d}\psi \\ &+ g(\alpha)\int_{\alpha}^{\pi} S(r,\phi,\psi)(\cos\alpha - \cos\psi)^{\kappa'}\,\mathrm{d}\psi \\ &+ \int_{\alpha}^{\beta} \mathrm{d}g(\theta)\int_{\theta}^{\pi} S(r,\phi,\psi)(\cos\theta) - \cos\psi)^{\kappa'}\,\mathrm{d}\psi\,.\end{aligned}$$

Proof. As in [3, Lemma 2], with $S(\phi,\psi)$ replaced by $S(r,\phi,\psi)$ which has a slightly different meaning, namely (6); and with κ replaced by κ', which here has the same meaning. □

Lemma 3. *For* $0 < r < 1$, $0 < \phi < \pi$, $0 < \psi < \pi$, *and* $S(r,\phi,\psi)$ *as defined in* (6),

$$S(r,\phi,\psi) = \sum_{n=-\infty}^{\infty} r^{|n|} v_n \cos v_n\phi \cos v_n(\pi - \psi) = \frac{\partial}{\partial\psi} T(r,\phi,\psi),$$

where

$$T(r,\phi,\psi) = \tfrac{1}{2}\{T(r,\psi+\phi) + T(r,\psi-\phi)\}$$

and

$$T(r,\chi) = \frac{1-r^2}{1+2r\cos\chi+r^2}\sin(v+\tfrac{1}{2})(\chi-\pi). \tag{8}$$

These functions T are all bounded for fixed r.

Proof. As in [3, Lemma 3] with $S(\phi,\psi)$ replaced by $S(r,\phi,\psi)$; effectively this is with r replaced by $-r$. □

Lemma 4. *If* $0<\alpha<\beta<\pi$, f *has bounded variation on* $[\cos\beta,\cos\alpha]$, $|\mathrm{re}\,\mu|<\frac{1}{2}$, $0<r<1$, J *is defined in* (7) *and* T *in* (8), *then*

$$J = J_1 - J_2 - J_3,$$

where

$$J_1(r,\phi) = g(\beta)\int_\beta^\pi T(r,\phi,\psi)\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\beta-\cos\psi)^{(1/2)-\mu}},$$

$$J_2(r,\phi) = g(\alpha)\int_\alpha^\pi T(r,\phi,\psi)\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\alpha-\cos\psi)^{(1/2)-\mu}},$$

$$J_3(r,\phi) = \int_\alpha^\beta \mathrm{d}g(\theta)\int_\theta^\pi T(r,\phi,\psi)\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\theta-\cos\psi)^{(1/2)-\mu}}.$$

Proof. As in [3, Lemma 4(i)] with κ replaced by $\frac{1}{2}+\mu$ and λ by $\frac{1}{2}-\mu$, and using the fact that $T(r,\phi,\pi)=0$ since by (8)

$$T(r,\pi\pm\phi) = \frac{1-r^2}{1-2r\cos\phi+r^2}\sin(v+\tfrac{1}{2})(\pm\phi). \qquad \square$$

3. Overview of subsequent procedure

In most of the paper my main aim is to prove that $B(r)$ in (2) tends to zero as $r\to 1-0$, supposing that $\alpha\leqslant\omega\leqslant\beta$. By (3), then, I have to prove that

$$I(r)\to 0 \quad \text{as } r\to 1-0,$$

$I(r)$ being given by (4)–(6). Considering absolute values, and writing $\kappa=\frac{1}{2}+\mathrm{re}\,\mu$ and $\lambda=\frac{1}{2}-\mathrm{re}\,\mu$, it is sufficient by (4), (7) and Lemma 4, to prove that

$$\int_0^\omega \frac{\mathrm{d}\phi}{(\cos\phi-\cos\omega)^\kappa}J_k(r,\phi)\to 0 \quad \text{as } r\to 1$$

for $k=1,2,3$. Note that κ and λ here are the real parts of the numbers denoted by κ and λ in [3].

For the case $k=3$ it is thus sufficient, by (8), to prove that both of the integrals

$$\int_0^\omega \frac{\mathrm{d}\phi}{(\cos\phi-\cos\omega)^\kappa}\int_\alpha^\beta \mathrm{d}g(\theta)\int_\theta^\pi T(r,\psi\pm\phi)\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\theta-\cos\psi)^\lambda}\to 0 \tag{9}$$

as $r\to 1$.

Similarly for the case $k=2$ it is sufficient to prove that both

$$g(\alpha)\int_0^{\omega}\frac{\mathrm{d}\phi}{(\cos\phi-\cos\omega)^{\kappa}}\int_{\alpha}^{\pi}T(r,\psi\pm\phi)\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\alpha-\cos\psi)^{\lambda}}\to 0 \tag{10}$$

as $r\to 1$; and for the case $k=1$ it is sufficient to prove that (10) holds with α replaced by β.

4. The integrals involving $T(r,\psi-\phi)$

From here on the work has little resemblance to that in [3].

Lemma 5. *If* $0<\varepsilon\leqslant\frac{1}{4}\pi$, $0<\kappa<1$, $0<\lambda<1$, $2\varepsilon\leqslant\omega\leqslant\pi-2\varepsilon$ *and* $2\varepsilon\leqslant\theta\leqslant\pi-2\varepsilon$, *then there are* K *and* L, *both independent of both* ω *and* θ, *such that*

$$\int_0^{\omega}\frac{\mathrm{d}\phi}{(\cos\phi-\cos\omega)^{\kappa}}\leqslant K \quad \textit{and} \quad \int_{\theta}^{\pi}\frac{\mathrm{d}\psi}{(\cos\theta-\cos\psi)^{\lambda}}\leqslant L.$$

Proof. (i) The ϕ-integral is equal to

$$\left(\int_0^{\varepsilon}+\int_{\varepsilon}^{\omega}\right)\frac{\mathrm{d}\phi}{(\cos\phi-\cos\omega)^{\kappa}}$$

$$\leqslant\int_0^{\varepsilon}\frac{\mathrm{d}\phi}{(\cos\varepsilon-\cos\omega)^{\kappa}}+\frac{1}{\sin\varepsilon}\int_{\varepsilon}^{\omega}\frac{\sin\phi\,\mathrm{d}\phi}{(\cos\phi-\cos\omega)^{\kappa}}$$

$$=\frac{\varepsilon}{(\cos\varepsilon-\cos\omega)^{\kappa}}+\frac{(\cos\varepsilon-\cos\omega)^{1-\kappa}}{(1-\kappa)\sin\varepsilon}$$

$$\leqslant\frac{\varepsilon}{(\cos\varepsilon-\cos 2\varepsilon)^{\kappa}}+\frac{(\cos\varepsilon-\cos 2\varepsilon)^{1-\kappa}}{(1-\kappa)\sin\varepsilon};$$

the last line is a possible value for K.

(ii) In (i) replace ω by $\pi-\theta$, ϕ by $\pi-\psi$ and κ by λ. Then (i) says that if $2\varepsilon\leqslant\theta\leqslant\pi-2\varepsilon$ then

$$\int_{\theta}^{\pi}\frac{\mathrm{d}\psi}{(\cos\theta-\cos\psi)^{\lambda}}\leqslant\frac{\varepsilon}{(\cos\varepsilon-\cos 2\varepsilon)^{\lambda}}+\frac{(\cos\varepsilon+\cos 2\varepsilon)^{1-\lambda}}{(1-\lambda)\sin\varepsilon};$$

this gives a possible value of L. □

Lemma 6. *If* $0<\varepsilon\leqslant\frac{1}{4}\pi$, T *is as in* (8), $0<\kappa<1$ *and* $0<\lambda<1$, *then as* $r\to 1$

$$\int_0^{\omega}\frac{\mathrm{d}\phi}{(\cos\phi-\cos\omega)^{\kappa}}\int_{\theta}^{\pi}|T(r,\psi-\phi)|\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\theta-\cos\psi)^{\lambda}}\to 0$$

uniformly with respect to θ *and* ω *in* $2\varepsilon\leqslant\theta\leqslant\pi-2\varepsilon$ *and* $2\varepsilon\leqslant\omega\leqslant\pi-2\varepsilon$.

Proof. (i) Let $0 < r < 1$ and $x = \phi - \psi + \pi$. By (8), remembering that v may be complex,

$$|T(r, \psi - \phi)| = \frac{1 - r^2}{1 + 2r\cos(\psi - \phi) + r^2}|\sin(v + \tfrac{1}{2})(\psi - \phi - \pi)|$$

$$= \frac{1 - r^2}{1 - 2r\cos x + r^2}|\sin(v + \tfrac{1}{2})x|. \tag{11}$$

Now

$$|\sin(v + \tfrac{1}{2})x| \leqslant \cosh \mathrm{i}m((v + \tfrac{1}{2})x)$$

$$= \cosh \mathrm{i}m(v(\phi - \psi + \pi))$$

$$\leqslant \cosh(2\pi\,\mathrm{i}mv) = m \quad \text{say,}$$

since in the double integral $|\phi - \psi + \pi| \leqslant 2\pi$. Thus m is independent of θ and ω, a fact which will be used occasionally.

Denote the double integral by I_-. Changing the variable in its inner integral, and using (11),

$$I_- = \int_0^\omega \frac{\mathrm{d}\phi}{(\cos\phi - \cos\omega)^\kappa} \int_\phi^{\phi+\pi-\theta} \frac{1 - r^2}{1 - 2r\cos x + r^2}|\sin(v + \tfrac{1}{2})x| \frac{\sin(x - \phi)\,\mathrm{d}x}{(\cos\theta + \cos(x - \phi))^\lambda}$$

$$= \int_0^{\pi-\theta+\omega} \frac{1 - r^2}{1 - 2r\cos x + r^2}|\sin(v + \tfrac{1}{2})x| h_-(x)\,\mathrm{d}x \tag{12}$$

where

$$h_-(x) = \int_{\max(0,\, x-\pi+\theta)}^{\min(\omega,\, x)} \frac{\sin(x - \phi)\,\mathrm{d}\phi}{(\cos\phi - \cos\omega)^\kappa(\cos\theta + \cos(x - \phi))^\lambda}; \tag{13}$$

the change of order of integration being valid because the integrand is nonnegative, noting that $0 \leqslant x - \phi = \pi - \psi \leqslant \pi$.

(ii) For $0 < x \leqslant \varepsilon \leqslant \frac{1}{2}\omega$ and $0 < \theta \leqslant \pi - 2\varepsilon$,

$$h_-(x) \leqslant \int_0^x \frac{\mathrm{d}\phi}{(\cos\phi - \cos\omega)^\kappa(\cos(\pi - 2\varepsilon) + \cos\varepsilon)^\lambda}$$

$$\leqslant \frac{1}{(\cos\varepsilon - \cos 2\varepsilon)^\lambda} \int_0^\omega \frac{\mathrm{d}\phi}{(\cos\phi - \cos\omega)^\kappa}$$

$$\leqslant \frac{K}{(\cos\varepsilon - \cos 2\varepsilon)^\lambda} = M \quad \text{say,}$$

using Lemma 5, so that M is independent of θ and ω.

(iii) No longer imposing the restriction $x \leqslant \varepsilon$, but only those in the enunciation of this lemma, (13) gives

$$\int_0^{\omega+\pi-\theta} h_-(x)\,\mathrm{d}x = \int_0^{\omega+\pi-\theta} \mathrm{d}x \int_{\max(0,\, x-\pi+\theta)}^{\min(\omega,\, x)} \frac{\sin(x - \phi)\,\mathrm{d}\phi}{(\cos\phi - \cos\omega)^\kappa(\cos\theta - \cos(x - \phi))^\lambda}$$

$$= \int_0^\omega \frac{d\phi}{(\cos\phi - \cos\omega)^\kappa} \int_\phi^{\phi+\pi-\theta} \frac{\sin(x-\phi)\, dx}{(\cos\theta + \cos(x-\phi))^\lambda}$$

$$= \int_0^\omega \frac{d\phi}{(\cos\phi - \cos\omega)^\kappa} \frac{(\cos\theta + 1)^{1-\lambda}}{1-\lambda} \leqslant \frac{2^{1-\lambda}}{1-\lambda} K$$

using Lemma 5. Thus h_- is integrable on $(0, \pi - \theta + \omega)$, with integral bounded by a number independent of θ and ω. Define $h_-(x) = 0$ for x outside $(0, \pi - \theta + \omega)$.

(iv) By (12), and (i)–(iii),

$$I_- = \left(\int_0^\varepsilon + \int_\varepsilon^{\pi/2} + \int_{\pi/2}^\pi \right) \frac{1-r^2}{1-2r\cos x + r^2} |\sin(v + \tfrac{1}{2})x| h_-(x)\, dx$$

$$\leqslant M \int_0^\varepsilon \frac{1-r^2}{1-2r\cos x + r^2} |\sin(v+\tfrac{1}{2})x|\, dx$$

$$+ (1-r^2) m \int_\varepsilon^{\pi/2} \frac{h_-(x)}{\sin^2 x}\, dx + (1-r^2) m \int_{\pi/2}^\pi \frac{h_-(x)}{1+r^2}\, dx$$

$$= o(1) + O(1-r^2) + O(1-r^2) = o(1) \quad \text{as } r \to 1,$$

the first o(1) coming from Poisson's integral and the whole expression being independent of θ and ω.

This completes the proof of Lemma 6. □

5. The integrals involving $T(r, \psi + \phi)$

In this section expressions analogous to I_- and h_- (in (12) and (13)) arise. It would be natural to denote them by I_+ and h_+, but because the treatment of them is so protracted I omit the subscript $+$. This involves some risk of confusion with $I(r)$ defined in (4), but the risk is minimal because $I(r)$ does not occur in this section.

Lemma 7. *If* $0 < \varepsilon \leqslant \frac{1}{4}\pi$, T *is as in* (8), $2\varepsilon \leqslant \theta \leqslant \pi - 2\varepsilon$, $2\varepsilon \leqslant \omega \leqslant \pi - 2\varepsilon$, $0 < \kappa < 1$, $0 < \lambda < 1$, *and* $x = \phi + \psi - \pi$ (*different from the* x *in Lemma* 6), *then* I_+ *defined by*

$$I_+ = \int_0^\omega \frac{d\phi}{(\cos\phi - \cos\omega)^\kappa} \int_\theta^\pi |T(r, \psi + \phi)| \frac{\sin\psi\, d\psi}{(\cos\theta - \cos\psi)^\lambda},$$

$$= \int_{\theta-\pi}^\omega \frac{1-r^2}{1-2r\cos x + r^2} |\sin(v+\tfrac{1}{2})x| h(x, \theta, \omega)\, dx \tag{14}$$

where

$$h(x, \theta, \omega) = \int_{\max(\theta, x+\pi-\omega)}^{\min(\pi, x+\pi)} \frac{\sin\psi\, d\psi}{(\cos\phi - \cos\omega)^\kappa (\cos\theta - \cos\psi)^\lambda}. \tag{15}$$

Further h, defined as zero for x outside $(\theta-\pi,\omega)$, is nonnegative and integrable with respect to x, and its integral has a bound independent of θ and ω.

Proof. (i) For fixed r in (0,1), (8) shows that T is bounded. It follows, using Lemma 5, that I is convergent. Also the order of integration can be changed, since the integrand is nonnegative. Thus

$$I=\int_\theta^\pi \frac{\sin\psi\,\mathrm{d}\psi}{(\cos\theta-\cos\psi)^\lambda}\int_0^\omega \frac{1-r^2}{1+2r\cos(\psi+\phi)+r^2}|\sin(v+\tfrac12)x|\frac{\mathrm{d}\phi}{(\cos\phi-\cos\omega)^\kappa}$$

$$=\int_\theta^\pi \frac{\sin\psi\,\mathrm{d}\psi}{(\cos\theta-\cos\psi)^\lambda}\int_{\psi-\pi}^{\omega+\psi-\pi} \frac{1-r^2}{1-2r\cos x+r^2}|\sin(v+\tfrac12)x|\frac{\mathrm{d}x}{(\cos\phi-\cos\omega)^\kappa}$$

$$=\int_{\theta-\pi}^\omega \frac{1-r^2}{1-2r\cos x+r^2}|\sin(v+\tfrac12)x|h(x,\theta,\omega)\,\mathrm{d}x$$

with $h(x,\theta,\omega)$ as defined in (15).

(ii) Defining $h(x,\theta,\omega)$ to be zero for x outside $(\theta-\pi,\omega)$,

$$\int_{-\pi}^\pi h(x,\theta,\omega)\,\mathrm{d}x=\int_{\theta-\pi}^\omega \mathrm{d}x\int_{\max(\theta,\,x+\pi-\omega)}^{\min(\pi,\,x+\pi)}\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\phi-\cos\omega)^\kappa(\cos\theta-\cos\psi)^\lambda}$$

$$=\int_\theta^\pi \frac{\sin\psi\,\mathrm{d}\psi}{(\cos\theta-\cos\psi)^\lambda}\int_{\psi-\pi}^{\omega+\psi-\pi}\frac{\mathrm{d}x}{(\cos\phi-\cos\omega)^\kappa}$$

$$=\int_\theta^\pi \frac{\sin\psi\,\mathrm{d}\psi}{(\cos\theta-\cos\psi)^\lambda}\int_0^\omega\frac{\mathrm{d}\phi}{(\cos\phi-\cos\omega)^\kappa}$$

$$\leqslant\frac{(\cos\theta+1)^{1-\lambda}}{1-\lambda}K\leqslant\frac{2^{1-\lambda}}{1-\lambda}K$$

by Lemma 5. This proves the rest of Lemma 7. □

Lemma 8. *If* $0<\varepsilon\leqslant\frac14\pi$, T *is as in* (8), $2\varepsilon\leqslant\theta\leqslant\pi-2\varepsilon$, $2\varepsilon\leqslant\omega\leqslant\pi-2\varepsilon$, $0<\kappa<1$ *and* $0<\lambda<1$, *then as* $r\to1$

$$I_+=\int_0^\omega\frac{\mathrm{d}\phi}{(\cos\phi-\cos\omega)^\kappa}\int_\theta^\pi|T(r,\psi+\phi)|\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\theta-\cos\psi)^\lambda}\to0$$

uniformly with respect to θ *and* ω *in* $\omega+\theta-\pi\leqslant-2\varepsilon$ *and in* $\omega+\theta-\pi\geqslant2\varepsilon$.

Proof. (i) Let $\omega+\theta-\pi\leqslant-2\varepsilon$. Also let $x=\phi+\psi-\pi$ and $-\varepsilon\leqslant x\leqslant\varepsilon$. Then, for the values of ϕ and ψ occurring in I,

$$\psi=\pi+x-\phi\geqslant\pi-\varepsilon-\omega,$$

$$\cos\psi\leqslant-\cos(\varepsilon+\omega)\leqslant-\cos(\pi-\theta-\varepsilon)=\cos(\theta+\varepsilon),$$

so that

$$\cos\theta - \cos\psi \geqslant \cos\theta - \cos(\theta+\varepsilon) = 2\sin(\theta+\tfrac{1}{2}\varepsilon)\sin\tfrac{1}{2}\varepsilon.$$

Further, since

$$\tfrac{1}{2}\varepsilon < \theta + \tfrac{1}{2}\varepsilon \leqslant \pi - \tfrac{3}{2}\varepsilon < \pi - \tfrac{1}{2}\varepsilon \quad \text{and} \quad \sin(\theta+\tfrac{1}{2}\varepsilon) > \sin\tfrac{1}{2}\varepsilon,$$

$$\cos\theta - \cos\psi \geqslant 2\sin^2\tfrac{1}{2}\varepsilon = 1-\cos\varepsilon.$$

Changing the variable in (15) by $\psi = \pi + x - \phi$,

$$0 \leqslant h(x,\theta,\omega) = \int_{\max(x,0)}^{\min(\pi+x-\theta,\,\omega)} \frac{\sin\psi\,\mathrm{d}\phi}{(\cos\phi-\cos\omega)^\kappa(\cos\theta-\cos\psi)^\lambda}$$

$$\leqslant \frac{1}{(1-\cos\varepsilon)^\lambda}\int_0^\omega \frac{\mathrm{d}\phi}{(\cos\phi-\cos\omega)^\kappa} \leqslant \frac{K}{(1-\cos\varepsilon)^\lambda}$$

using the last inequality in the previous paragraph and Lemma 5. Now by (14) in Lemma 7,

$$I \leqslant \int_{-\pi+\varepsilon}^{\pi-\varepsilon} \frac{1-r^2}{1-2r\cos x+r^2}|\sin(v+\tfrac{1}{2})x|h(x,\theta,\omega)\,\mathrm{d}x$$

$$\leqslant \frac{K}{(1-\cos\varepsilon)^\lambda}\int_{-\varepsilon}^{\varepsilon} \frac{1-r^2}{1-2r\cos x+r^2}|\sin(v+\tfrac{1}{2})x|\,\mathrm{d}x$$

$$+\left(\int_{-\pi+\varepsilon}^{-\varepsilon} + \int_{\varepsilon}^{\pi-\varepsilon}\right)\frac{1-r^2}{\sin^2 x} m h(x,\theta,\omega)\,\mathrm{d}x$$

$$\leqslant \mathrm{o}(1) + \frac{1-r^2}{\sin^2\varepsilon} m \int_{-\pi}^{\pi} h(x,\theta,\omega)\,\mathrm{d}x$$

$$= \mathrm{o}(1) + \mathrm{O}(1-r^2) = \mathrm{o}(1) \quad \text{as } r \to 1;$$

here the first o(1) comes from Poisson's integral, and the $\mathrm{O}(1-r^2)$ from the last sentence in Lemma 7, m being defined in (i) of Lemma 6. All these inequalities are independent of θ and ω in $\omega+\theta-\pi \leqslant -2\varepsilon$ (together with $2\varepsilon \leqslant \omega \leqslant \pi-2\varepsilon$ and $2\varepsilon \leqslant \theta \leqslant \pi - 2\varepsilon$).

Thus $I \to 0$ as $r \to 1$ uniformly on $\omega+\theta-\pi \leqslant -2\varepsilon$.

(ii) Let $\omega+\theta-\pi \geqslant 2\varepsilon$. Also let $x = \phi+\psi-\pi$ and $-\varepsilon \leqslant x \leqslant \varepsilon$. Since $\psi \geqslant \theta$ in I, $\phi = \pi + x - \psi \leqslant \pi + \varepsilon - \theta$,

$$\cos\phi \geqslant \cos(\pi+\varepsilon-\theta) = -\cos(\theta-\varepsilon) \geqslant -\cos(\pi-\omega+\varepsilon) = \cos(\omega-\varepsilon),$$

so that

$$\cos\phi - \cos\omega \geqslant \cos(\omega-\varepsilon) - \cos\omega = 2\sin(\omega-\tfrac{1}{2}\varepsilon)\sin\tfrac{1}{2}\varepsilon.$$

Further, since $\tfrac{1}{2}\varepsilon < \omega - \tfrac{1}{2}\varepsilon < \pi - \tfrac{1}{2}\varepsilon$, $\sin(\omega-\tfrac{1}{2}\varepsilon) \geqslant \sin\tfrac{1}{2}\varepsilon$; thus

$$\cos\phi - \cos\omega \geqslant 2\sin^2\tfrac{1}{2}\varepsilon = 1-\cos\varepsilon.$$

By (15),

$$h(x,\theta,\omega) = \int_{\max(\theta,\,x+\pi-\omega)}^{\min(\pi,\,x+\pi)} \frac{\sin\psi\,\mathrm{d}\psi}{(\cos\phi-\cos\omega)^\kappa(\cos\theta-\cos\psi)^\lambda}$$

$$\leqslant \frac{1}{(1-\cos\varepsilon)^{\kappa}} \int_{\theta}^{\pi} \frac{\sin\psi\,\mathrm{d}\psi}{(\cos\theta-\cos\psi)^{\lambda}}$$

$$= \frac{1}{(1-\cos\varepsilon)^{\kappa}} \frac{(\cos\theta+1)^{1-\lambda}}{1-\lambda} \leqslant \frac{2^{1-\lambda}}{(1-\cos\varepsilon)^{\kappa}(1-\lambda)}.$$

Thus h is bounded on $-\varepsilon \leqslant x \leqslant \varepsilon$ independently of θ and ω.

It follows as in the latter part of (i) that $I \to 0$ as $r \to 1$ uniformly on $\omega+\theta-\pi \geqslant 2\varepsilon$, thus completing the proof of Lemma 8. □

Lemma 9. *If h is as in* (15) *and h^* is defined by*

$$h^*(x,\theta,\omega) := \int_{\max(\theta,\, x+\pi-\omega)}^{\min(\pi,\, x+\pi)} \frac{\mathrm{d}\psi}{(\cos\phi-\cos\omega)^{\kappa}(\cos\theta-\cos\psi)^{\lambda}}, \tag{16}$$

and if $0<\varepsilon \leqslant \frac{1}{4}\pi$, $2\varepsilon \leqslant \theta \leqslant \pi-2\varepsilon$, $2\varepsilon \leqslant \omega \leqslant \pi-2\varepsilon$, $0<\kappa<1=\kappa+\lambda$, $\rho=\min(\kappa,\lambda)$, $\phi = x+\pi-\psi$, $0 \leqslant \theta+\omega-\pi \leqslant 2\varepsilon$ *and* $0<x\leqslant\varepsilon$, *then*

$$0 \leqslant xh(x,\theta,\omega) \leqslant xh^*(x,\theta,\omega) = \mathrm{O}(x^{\rho}) \quad \text{as } x \to 0+$$

where O *is independent of θ and ω.*

Proof. (i) Since $\theta<\pi<\pi+x$ and $x+\pi-\omega \leqslant \varepsilon+\pi-2\varepsilon<\pi$,

$$\max(\theta, x+\pi-\omega) < \min(\pi, x+\pi).$$

So $0 \leqslant h \leqslant h^*$. It remains to show that $h^* = \mathrm{O}(x^{\rho-1})$ as $x \to 0+$.

(ii) Let $\sigma = \theta+\omega-\pi$. Suppose that $0<x\leqslant \frac{1}{2}\sigma \leqslant \varepsilon$ for the present. By (16),

$$h^* \leqslant \int_{\theta}^{\pi} \frac{\mathrm{d}\psi}{(\cos\phi-\cos\omega)^{\kappa}(\cos\theta-\cos\psi)^{\lambda}}$$

$$= \int_{\theta}^{\pi} \frac{\mathrm{d}\psi}{(\cos(\pi-\omega)-\cos(\psi-x))^{\kappa}(\cos\theta-\cos\psi)^{\lambda}}$$

$$\leqslant \frac{1}{(\cos(\pi-\omega)-\cos(\theta-x))^{\kappa}} \int_{\theta}^{\pi} \frac{\mathrm{d}\psi}{(\cos\theta-\cos\psi)^{\lambda}}$$

$$\leqslant L/(2\sin\tfrac{1}{2}(\theta-x+\pi-\omega)\sin\tfrac{1}{2}(\theta-x-\pi+\omega))^{\kappa}$$

$$= L/(2\cos\tfrac{1}{2}(x+\omega-\theta)\sin\tfrac{1}{2}(\sigma-x))^{\kappa},$$

where L arises from Lemma 5. But

$$\tfrac{1}{2}(x+\omega-\theta) \leqslant \tfrac{1}{2}(\varepsilon+\pi-2\varepsilon-2\varepsilon) < \tfrac{1}{2}\pi-\varepsilon$$

and

$$\tfrac{1}{2}(x+\omega-\theta) > \tfrac{1}{2}(\omega-\theta) \geqslant \tfrac{1}{2}(2\varepsilon-\pi+2\varepsilon) > -\tfrac{1}{2}\pi+\varepsilon,$$

so that

$$\cos\tfrac{1}{2}(x+\omega-\theta) \geqslant \cos(\tfrac{1}{2}\pi-\varepsilon) = \sin\varepsilon.$$

Also

$$\sin\tfrac{1}{2}(\sigma - x) \geqslant \sin\tfrac{1}{2}(2x - x) = \sin\tfrac{1}{2}x > x/\pi.$$

Thus, observing that $0 < x < \varepsilon < 1$,

$$xh^*(x,\theta,\omega) \leqslant xL\left(\frac{\pi/x}{2\sin\varepsilon}\right)^{\kappa} = \mathrm{O}(x^{\lambda}) = \mathrm{O}(x^{\rho})$$

with O independent of θ and ω.

(iii) Now suppose instead that $\sigma = 0 < x \leqslant \varepsilon$. Then $\max(\theta, x + \pi - \omega) = \max(\theta, x + \theta) = x + \theta \leqslant \varepsilon + \pi - 2\varepsilon < \min(\pi, x + \pi)$, so that, by (16),

$$\begin{aligned} h^*(x,\theta,\omega) &= \int_{x+\theta}^{\pi} \frac{\mathrm{d}\psi}{(\cos\phi - \cos\omega)^{\kappa}(\cos\theta - \cos\psi)^{\lambda}} \\ &= \int_{x}^{\pi-\theta} \frac{\mathrm{d}\phi}{(\cos\phi - \cos\omega)^{\kappa}(\cos\theta - \cos(\pi - \phi + x))^{\lambda}} \\ &\leqslant \frac{1}{(\cos\theta - \cos(\theta + x))^{\lambda}} \int_{x}^{\omega} \frac{\mathrm{d}\phi}{(\cos\phi - \cos\omega)^{\kappa}} \\ &\leqslant \frac{K}{(2\sin(\theta + \frac{1}{2}x)\sin\frac{1}{2}x)^{\lambda}} \quad \text{using Lemma 5.} \end{aligned}$$

But

$$\varepsilon < 2\varepsilon \leqslant \theta < \theta + \tfrac{1}{2}x \leqslant \pi - 2\varepsilon + \tfrac{1}{2}\varepsilon < \pi - \varepsilon,$$

so that

$$\sin(\theta + \tfrac{1}{2}x) > \sin\varepsilon;$$

also

$$\sin\tfrac{1}{2}x > x/\pi.$$

Thus

$$xh^*(x,\theta,\omega) \leqslant \left(\frac{\pi}{2\sin\varepsilon}\right)^{\lambda} \frac{Kx}{x^{\lambda}} = \left(\frac{\pi}{2\sin\varepsilon}\right)^{\lambda} Kx^{\kappa} = \mathrm{O}(x^{\rho})$$

with O independent of θ and ω. □

Lemma 10. *If h is as in* (15) *and h^* as in* (16), $0 < \varepsilon \leqslant \frac{1}{4}\pi$, $2\varepsilon \leqslant \theta \leqslant \pi - 2\varepsilon$, $2\varepsilon \leqslant \omega \leqslant \pi - 2\varepsilon$, $0 < \kappa < 1 = \kappa + \lambda$, $\rho = \min(\kappa, \lambda)$, $\phi = x + \pi - \psi$, $0 \leqslant \theta + \omega - \pi \leqslant 2\varepsilon$ *and* $-\varepsilon \leqslant x < 0$, *then*

$$0 \leqslant -xh(x,\theta,\omega) \leqslant -xh^*(x,\theta,\omega) = \mathrm{O}(|x|^{\rho}) \quad \text{as } x \to 0-$$

where O *is independent of θ and ω.*

Proof. Let $\sigma = \theta + \omega - \pi$ and $y = -x$, so that $0 \leqslant \sigma \leqslant 2\varepsilon$ and $0 < y \leqslant \varepsilon$. Since

$$x + \pi - \omega = \theta - \sigma - y < \theta \quad \text{and} \quad \pi + x \geqslant \pi - \varepsilon > \pi - 2\varepsilon \geqslant \theta,$$

$$\max(\theta, x + \pi - \omega) = \theta < \min(\pi + x, \pi) = \pi - y;$$

thus by (16)

$$\begin{aligned} 0 \leqslant h^*(x,\theta,\omega) &= \int_\theta^{\pi-y} \frac{\mathrm{d}\psi}{(\cos(\pi-\omega) - \cos(\psi+y))^\kappa (\cos\theta - \cos\psi)^\lambda} \\ &\leqslant \frac{1}{(\cos(\pi-\omega) - \cos(\theta+y))^\kappa} \int_\theta^{\pi-y} \frac{\mathrm{d}\psi}{(\cos\theta - \cos\psi)^\lambda} \\ &\leqslant \frac{L}{(2\sin\frac{1}{2}(\theta + y + \pi - \omega)\sin\frac{1}{2}(\theta + y - \pi + \omega))^\kappa} \quad \text{using Lemma 5,} \\ &\leqslant \frac{L}{(2\cos\frac{1}{2}(\omega - \theta - y)\sin\frac{1}{2}(\sigma + y))^\kappa}. \end{aligned}$$

Now

$$\tfrac{1}{2}(\omega - \theta - y) \geqslant \tfrac{1}{2}(2\varepsilon - \kappa + 2\varepsilon - \varepsilon) > -\tfrac{1}{2}\pi + \varepsilon$$

and

$$\tfrac{1}{2}(\omega - \theta - y) \leqslant \tfrac{1}{2}\omega \leqslant \tfrac{1}{2}\pi - \varepsilon,$$

so that

$$\cos\tfrac{1}{2}(\omega - \theta - y) \geqslant \cos(\tfrac{1}{2}\pi - \varepsilon) = \sin\varepsilon.$$

Also

$$0 < \tfrac{1}{2}y \leqslant \tfrac{1}{2}(\sigma + y) \leqslant \tfrac{1}{2}(2\varepsilon + \varepsilon) < 2\varepsilon \leqslant \tfrac{1}{2}\pi,$$

so that

$$\sin\tfrac{1}{2}(\sigma + y) > (\sigma + y)/\pi \geqslant y/\pi.$$

Thus

$$|x| h^*(x,\theta,\omega) \leqslant \left(\frac{\pi}{2\sin\varepsilon}\right)^\kappa \frac{Ly}{y^\kappa} = \left(\frac{\pi}{2\sin\varepsilon}\right)^\kappa L y^\lambda = \mathrm{O}(|x|^\rho)$$

with O independent of θ and ω. This proves Lemma 10. □

Lemma 11. *If h is as in* (15) *and h^* as in* (16), $0 < \varepsilon \leqslant \frac{1}{4}\pi$, $2\varepsilon \leqslant \theta \leqslant \pi - 2\varepsilon$, $2\varepsilon \leqslant \omega \leqslant \pi - 2\varepsilon$, $0 < \kappa < 1 = \kappa + \lambda$, $\rho = \min(\kappa, \lambda)$, $\phi = x + \pi - \psi$, $-2\varepsilon \leqslant \theta + \omega - \pi < 0$ *and* $-\varepsilon \leqslant x \leqslant \varepsilon$, *then*

$$0 \leqslant |x| h(x,\theta,\omega) \leqslant |x| h^*(x,\theta,\omega) = \mathrm{O}(|x|^\rho) \quad \text{as } x \to 0$$

where O *is independent of* θ *and* ω.

Remark. The only difference between the hypotheses of Lemmas 9 and 10 is that x is positive in one and negative in the other. Together those two lemmas imply that if $0 \leqslant \theta + \omega - \pi \leqslant 2\varepsilon$ (and the other hypotheses) then

$$|x| h^*(x, \theta, \omega) = \mathrm{O}(|x|^\rho) \quad \text{as } x \to 0.$$

In Lemma 11, I aim to prove the same conclusion when $\theta + \omega - \pi$ is in $[-2\varepsilon, 0)$ instead of in $[0, 2\varepsilon]$.

Proof. Attach the suffix 1 to the symbols $x, \theta, \omega, \kappa, \lambda, \phi, \psi$ in the present hypothesis.
Let

$$\begin{aligned} &x_2 = -x_1, \quad \theta_2 = \pi - \omega_1, \quad \omega_2 = \pi - \theta_1, \quad \kappa_2 = \lambda_1, \quad \lambda_2 = \kappa_1, \\ &\phi_2 = \pi - \psi_1, \quad \psi_2 = \pi - \phi_1. \end{aligned} \tag{17}$$

I aim to obtain an estimate for

$$h^* = h^*(x_1, \theta_1, \omega_1) = \int_{\max(\theta_1, x_1+\pi-\omega_1)}^{\min(\pi, x_1+\pi)} \frac{\mathrm{d}\psi_1}{(\cos\phi_1 - \cos\omega_1)^{\kappa_1}(\cos\theta_1 - \cos\psi_1)^{\lambda_1}}.$$

Translating this integral into terms of the symbols with suffix 2 by means of (17),

$$h^* = \int_{\max(\pi-\omega_2,\, \theta_2-x_2)}^{\min(\pi,\, \pi-x_2)} \frac{\mathrm{d}(\pi - \phi_2)}{(\cos\theta_2 - \cos\psi_2)^{\lambda_2}(\cos\phi_2 - \cos\omega_2)^{\kappa_2}}.$$

Now $\phi_2 + \psi_2 = 2\pi - \psi_1 - \phi_1 = \pi - x_1 = \pi + x_2$, so that $\phi_2 = x_2 + \pi - \psi_2$; thus

$$h^* = \int_{\max(0, x_2)}^{\min(\omega_2,\, \pi+x_2-\theta_2)} \frac{\mathrm{d}\phi_2}{(\cos\phi_2 - \cos\omega_2)^{\kappa_2}(\cos\theta_2 - \cos\psi_2)^{\lambda_2}}.$$

Also $\phi_2 = \min(\omega_2, \pi + x_2 - \theta_2)$ makes $\psi_2 = \pi + x_2 + \max(-\omega_2, -\pi - x_2 + \theta_2)$, and $\phi_2 = \max(0, x_2)$ makes $\psi_2 = \pi + x_2 + \min(0, -x_2) = \min(\pi + x_2, \pi)$; so that

$$h^* = \int_{\max(\pi+x_2-\omega_2, \theta_2)}^{\min(\pi+x_2, \pi)} \frac{\mathrm{d}\psi_2}{(\cos\phi_2 - \cos\omega_2)^{\kappa_2}(\cos\theta_2 - \cos\psi_2)^{\lambda_2}}. \tag{18}$$

Now by (17) the symbols with suffix 2 satisfy the hypotheses of Lemmas 9 and 10; so these lemmas give that the right-hand side of (18) is $\mathrm{O}(|x|^{\rho-1})$ as $x \to 0$, whence

$$|x| h^* = \mathrm{O}(|x|^\rho) \quad \text{as } x \to 0$$

with O independent of θ and ω, as required for Lemma 11. □

Lemma 12. *If* $2\varepsilon \leqslant \alpha \leqslant \pi - \omega - 2\varepsilon < \pi - \omega + 2\varepsilon \leqslant \beta \leqslant \pi - 2\varepsilon$, *g has bounded variation on* $\alpha \leqslant \theta \leqslant \beta$, $2\varepsilon \leqslant \omega \leqslant \pi - 2\varepsilon, 0 < \kappa < 1 = \kappa + \lambda$ *and* $\rho = \min(\kappa, \lambda)$, *then*

$$\int_\alpha^\beta |\mathrm{d}g(\theta)| \int_0^\omega \frac{\mathrm{d}\phi}{(\cos\phi - \cos\omega)^\kappa} \int_\theta^\pi |T(r, \psi \pm \phi)| \frac{\sin\psi\, \mathrm{d}\psi}{(\cos\theta - \cos\psi)^\lambda} \to 0$$

as $r \to 1$, *g being defined in* (5) *and* T *in* (8).

Proof. Denoting these two triple integrals by $J_\pm$, the inner double integral in J_- tends to zero as $r \to 1$, uniformly on $\alpha \leqslant \theta \leqslant \beta$ by Lemma 6. So J_- itself tends to zero as $r \to 1$, since $g(\theta) = f(\cos\theta)\sin^{-\mu}\theta$ has bounded variation on that interval.

It remains to show that J_+ tends to zero as $r \to 1$. Denoting the inner double integral in J_+ by $I_+(r,\theta)$, and writing α' and β' for $\pi - \omega - 2\varepsilon$ and $\pi - \omega + 2\varepsilon$ respectively,

$$J_+ = \left(\int_\alpha^{\alpha'} + \int_{\alpha'}^{\beta'} + \int_{\beta'}^{\beta}\right) I_+(r,\theta)|\,\mathrm{d}g(\theta)|. \tag{19}$$

The first and third of these integrals tend to zero as $r \to 1$, by the uniform convergence of I_+ established in Lemma 8. The second of these integrals is expressible, by Lemma 7, as

$$\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos x + r^2}\,\mathrm{d}x \int_{\alpha'}^{\beta'} |\sin(v+\tfrac{1}{2})x|h(x,\theta,\omega)\,|\,\mathrm{d}g(\theta)| \tag{20}$$

after changing the order of integration. The inner integral here is

$$\left|\frac{\sin(v+\frac{1}{2})x}{x}\right| \int_{\alpha'}^{\beta'} |x|h(x,\theta,\omega)|\,\mathrm{d}g(\theta)| = \mathrm{O}(1)\int_{\pi-\omega-2\varepsilon}^{\pi-\omega+2\varepsilon} \mathrm{O}(|x|^\rho)|\,\mathrm{d}g(\theta)|$$

as $x \to 0$, by Lemmas 9–11, with O independent of θ. Thus (20) is

$$\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos x + r^2}\mathrm{O}(|x|^\rho)\,\mathrm{d}x.$$

By Poisson's integral, if applicable, this gives that (20) tends to zero as $r \to 1$. For this applicability, integrability of the inner integral in (20) is sufficient; and for reassurance of this,

$$\begin{aligned}
&\int_{-\pi}^{\pi} \mathrm{d}x \int_{\alpha'}^{\beta'} |\sin(v+\tfrac{1}{2})x|h(x,\theta,\omega)|\,\mathrm{d}g(\theta)| \\
&= \int_{\alpha'}^{\beta'} |\,\mathrm{d}g(\theta)| \int_{-\pi}^{\pi} |\sin(v+\tfrac{1}{2})x|h(x,\theta,\omega)\,\mathrm{d}x \\
&\leqslant m \int_{\alpha'}^{\beta'} |\,\mathrm{d}g(\theta)| \int_{-\pi}^{\pi} h(x,\theta,\omega)\,\mathrm{d}x
\end{aligned}$$

with m as in Lemma 6(i). This last expression is finite by Lemma 7, giving the required applicability of Poisson's integral. This completes the proof of Lemma 12. □

6. An Abel-summable series

Note. From here on x denotes a constant, $\cos\omega$; it has no connection with the variable x introduced in Lemma 7 and used extensively in the subsequent work to Lemma 12. This change of notation is made in order to conform to the use of x in [2,3].

Lemma 13. *Let f be integrable on its support* $[\cos\beta, \cos\alpha]$, *and let the requirements of Lemma* 12 *hold with*

$$2\varepsilon \leqslant \alpha \leqslant \pi - \omega - 2\varepsilon < \pi - \omega + 2\varepsilon \leqslant \beta < \pi - 2\varepsilon.$$

If $|\mathrm{re}\,\mu| < \frac{1}{2}$, v *is not half an odd integer*, $x = \cos\omega$ *and, as in* (1),

$$b_n = \frac{v+n+\frac{1}{2}}{2\cos v\pi} \int_{-1}^{1} f(t)\mathrm{P}_{v+n}^{-\mu}(-t)\,\mathrm{d}t$$

then

$$b_0 \mathrm{P}_v^{\mu}(x) + \sum_{n=1}^{\infty} r^n \{b_n \mathrm{P}_{v+n}^{\mu}(x) + b_{-n}\mathrm{P}_{v-n}^{\mu}(x)\} \to 0 \quad \text{as } r \to 1-.$$

Proof. By (2) and (3) in Lemma 1 it is enough to prove that $I(r) \to 0$ as $r \to 1-$, where $I(r)$ is defined in (4)–(6) with $\alpha \geqslant 2\varepsilon$ and $\beta \leqslant \pi - 2\varepsilon$. By (7) in Lemma 2,

$$I(r) = \int_0^{\omega} \frac{J(r,\phi)}{(\cos\phi - \cos\omega)^{(1/2)+\mu}}\,\mathrm{d}\phi$$

where $J = J_1 - J_2 - J_3$ is given in Lemma 4 in terms of

$$T(r,\phi,\psi) = \tfrac{1}{2}\{T(r,\psi+\phi) + T(r,\psi-\phi)\},$$

which is defined at (8) in Lemma 3. It is therefore sufficient to prove that all six of

$$g(\alpha)\int_0^{\omega} \frac{\mathrm{d}\phi}{(\cos\phi - \cos\omega)^{(1/2)+\mu}} \int_{\alpha}^{\pi} T(r,\psi\pm\phi)\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\alpha - \cos\psi)^{(1/2)-\mu}}, \tag{21$\pm$}$$

$$g(\beta)\int_0^{\omega} \frac{\mathrm{d}\phi}{(\cos\phi - \cos\omega)^{(1/2)+\mu}} \int_{\beta}^{\pi} T(r,\psi\pm\phi)\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\beta - \cos\psi)^{(1/2)-\mu}}, \tag{22$\pm$}$$

$$\int_{\alpha}^{\beta} \mathrm{d}g(\theta)\int_0^{\omega} \frac{\mathrm{d}\phi}{(\cos\phi - \cos\omega)^{(1/2)+\mu}} \int_{\theta}^{\pi} T(r,\psi\pm\phi)\frac{\sin\psi\,\mathrm{d}\psi}{(\cos\theta - \cos\psi)^{(1/2)-\mu}} \tag{23$\pm$}$$

tend to zero as $r \to 1$. This will be done after replacing the integrands by their absolute values; that is, replacing $\mathrm{d}g(\theta)$ by $|\mathrm{d}g(\theta)|$, $T(r,\psi\pm\phi)$ by $|T(r,\psi\pm\phi)|$, $\frac{1}{2}+\mu$ by $\kappa = \frac{1}{2} + \mathrm{re}\,\mu$ and $\frac{1}{2}-\mu$ by $\lambda = \frac{1}{2} - \mathrm{re}\,\mu$. By hypothesis κ and λ lie in $(0,1)$. Since the integrands thereby become nonnegative the orders of integration can be changed.

Lemma 6 gives that (21−) and (22−) tend to zero as $r \to 1$. It also gives that (23−) tends to zero as $r \to 1$ because the inner double integral therein tends to zero uniformly with respect to θ in $[\alpha, \beta]$.

Lemma 8 gives that (21+) and (22+) tend to zero as $r \to 1$, since $\alpha \leqslant \pi - \omega - 2\varepsilon$ and $\pi - \omega + 2\varepsilon \leqslant \beta$.

Lemma 12 gives that (23+) tends to zero as $r \to 1$. This completes the proof of Lemma 13. □

7. The "unlocalized" expansion of zero

Until now the function f has been supported on a closed subinterval of $(-1,1)$. This restriction will now be removed.

Definition. As in [2, (2.2)], let

$$E(v,t)=E(\mu,v,x,t)$$
$$=\tfrac{1}{2}\{(v-\mu)\mathrm{P}_v^{\mu}(x)\mathrm{P}_{v-1}^{-\mu}(-t)-(v+\mu)\mathrm{P}_{v-1}^{\mu}(x)\mathrm{P}_v^{-\mu}(-t)\}. \tag{24}$$

Lemma 14. *If* $(1-t^2)^{-1/4}f(t)\in L(-1,1)$, f *vanishes in a neighbourhood of a certain* $-x\in(-1,1)$, $|\mathrm{re}\,\mu|<\frac{1}{2}$, v *is not half an odd integer, and* b_n *is as in* (1) (*and as in Lemma* 13), *then*

$$2\cos v\pi\sum_{r=0}^{n-1} b_r\mathrm{P}_{v+r}^{\mu}(x)=\int_{-1}^{1}\frac{f(t)}{x+t}E(v+n,t)\,\mathrm{d}t-\int_{-1}^{1}\frac{f(t)}{x+t}E(v,t)\,\mathrm{d}t,$$

the integrals being convergent Lebesgue integrals.

Proof. For $t\neq -x$, [2, Theorem 1, (2.5) and (2.3)] give, replacing m by 1 and n by $n-1$,

$$\sum_{r=0}^{n-1}(v+r+\tfrac{1}{2})\mathrm{P}_{v+r}^{\mu}(x)\mathrm{P}_{v+r}^{-\mu}(-t)=\frac{E(v+n,t)-E(v,t)}{x+t}.$$

Integrating with respect to $f(t)\,\mathrm{d}t$,

$$2\cos v\pi\sum_{r=0}^{n-1} b_r\mathrm{P}_{v+r}^{\mu}(x)=\sum_{r=0}^{n-1}(v+r+\tfrac{1}{2})\mathrm{P}_{v+r}^{\mu}(x)\int_{-1}^{1}\mathrm{P}_{v+r}^{-\mu}(-t)f(t)\,\mathrm{d}t$$
$$=\int_{-1}^{1}\frac{f(t)}{x+t}E(v+n,t)\,\mathrm{d}t-\int_{-1}^{1}\frac{f(t)}{x+t}E(v,t)\,\mathrm{d}t,$$

all integrals occurring existing in the Lebesgue sense, for the following reasons. The integrals occurring in the b_r exist by [2, Theorem 2] with μ replaced by $-\mu$. The two integrals involving E need the following discussion.

Choose positive δ such that $f(t)=0$ for $-x-\delta\leqslant t\leqslant -x+\delta$ and such that this interval is contained in $-1<t<1$. Then

$$\begin{aligned}&(x+t)^{-1}f(t)=0\quad\text{in } -x-\delta\leqslant t<-x\quad\text{and}\quad -x<t\leqslant -x+\delta,\\&|(x+t)^{-1}f(t)|\leqslant\delta^{-1}|f(t)|\quad\text{in } -1<t<-x-\delta\quad\text{and}\quad -x+\delta<t<1.\end{aligned}\tag{25}$$

For fixed μ and v, and $-t=\cos\theta$, [2, Theorem 3] gives that

$$\mathrm{P}_v^{-\mu}(-t)=\mathrm{P}_v^{-\mu}(\cos\theta)=\mathrm{O}(\sin^{-1/2}\theta)=\mathrm{O}((1-t^2)^{-1/4})$$

with O independent of θ in $(0,\pi)$, and therefore of t in $(-1,1)$. Similarly $\mathrm{P}_{v-1}^{-\mu}(-t)=\mathrm{O}((1-t^2)^{-1/4})$.

Now $E(v,t)$ is a linear combination of $\mathrm{P}_{v-1}^{-\mu}(-t)$ and $\mathrm{P}_v^{-\mu}(-t)$, so that

$$E(v,t)=\mathrm{O}((1-t^2)^{-1/4})\quad\text{in } -1<t<1.$$

Thus by (25)

$$|(x+t)^{-1}f(t)E(v,t)|=\mathrm{O}(|f(t)|(1-t^2)^{-1/4})\in L(-1,1).$$

Similarly $(x+t)^{-1}f(t)E(v+n,t)\in L(-1,1)$ since the integer n is fixed in this lemma. This completes the proof of Lemma 14.

Lemma 15. *Under the hypotheses of Lemma* 14,

$$\sum_{r=0}^{\infty} b_r \mathrm{P}_{\nu+r}^{\mu}(x) = -\frac{1}{2\cos\nu\pi}\int_{-1}^{1}\frac{f(t)}{x+t}E(\nu,t)\,\mathrm{d}t$$

and

$$\sum_{r=1}^{\infty} b_{-r} \mathrm{P}_{\nu-r}^{\mu}(x) = +\frac{1}{2\cos\nu\pi}\int_{-1}^{1}\frac{f(t)}{x+t}E(\nu,t)\,\mathrm{d}t.$$

Proof. (i) For the first equation it is sufficient, by Lemma 14, to prove that

$$\int_{-1}^{1}\frac{f(t)}{x+t}E(\nu+n,t)\,\mathrm{d}t \to 0 \quad \text{as } n\to\infty.$$

By (25), $|(x+t)^{-1}f(t)(1-t^2)^{-1/4}| \leqslant \delta^{-1}|f(t)(1-t^2)^{-1/4}| \in L(-1,1)$; whence by [2, Theorem 4] with μ and ν replaced by $-\mu$ and $\nu+n$, respectively,

$$\int_{-1}^{1}\frac{f(t)}{x+t}\mathrm{P}_{\nu+n}^{-\mu}(-t)\,\mathrm{d}t = \mathrm{o}(n^{-\mathrm{re}\,\mu-(1/2)}) \quad \text{as } n\to\infty.$$

By [2, Theorem 3] with θ fixed and ν replaced by $\nu+n-1$,

$$\mathrm{P}_{\nu+n-1}^{\mu}(x) = \mathrm{O}(n^{\mathrm{re}\,\mu-(1/2)}) \quad \text{as } n\to\infty.$$

Thus

$$(\nu+n+\mu)\mathrm{P}_{\nu+n-1}^{\mu}(x)\int_{-1}^{1}\frac{f(t)}{x+t}\mathrm{P}_{\nu+n}^{-\mu}(-t)\,\mathrm{d}t$$

$$= \mathrm{O}(n)\mathrm{O}(n^{\mathrm{re}\,\mu-(1/2)})\mathrm{o}(n^{-\mathrm{re}\,\mu-(1/2)}) = \mathrm{o}(1).$$

By reasoning closely parallel to that of the last paragraph,

$$(\nu+n-\mu)\mathrm{P}_{\nu+n}^{\mu}(x)\int_{-1}^{1}\frac{f(t)}{x+t}\mathrm{P}_{\nu+n-1}^{-\mu}(-t)\,\mathrm{d}t = \mathrm{o}(1).$$

Subtracting these last two equations, and observing (24),

$$\int_{-1}^{1}\frac{f(t)}{x+t}E(\nu+n,t)\,\mathrm{d}t = \mathrm{o}(1) \quad \text{as } n\to\infty.$$

This with Lemma 14 proves the first equation of Lemma 15.

(ii) The second equation will be deduced from the first by several applications of [1, 3.4(7)], which is

$$\mathrm{P}_{\nu}^{\mu}(x) = \mathrm{P}_{-\nu-1}^{\mu}(x).$$

Writing $b_r(\nu)$ as a fuller notation for b_r,

$$b_{-r-1}(\nu) = \frac{\nu-r-\frac{1}{2}}{2\cos\nu\pi}\int_{-1}^{1} f(t)\mathrm{P}_{\nu-r-1}^{-\mu}(-t)\,\mathrm{d}t$$

$$= -\frac{-\nu+r+\frac{1}{2}}{2\cos(-\nu\pi)}\int_{-1}^{1} f(t)\mathrm{P}_{-\nu+r}^{\mu}(-t)\,\mathrm{d}t = -b_r(-\nu);$$

this with (i) gives

$$\sum_{r=0}^{\infty} b_{-r-1}(v)\mathrm{P}^{\mu}_{v-r-1}(x) = -\sum_{r=0}^{\infty} b_r(-v)\mathrm{P}^{\mu}_{-v+r}(x) = \frac{1}{2\cos v\pi}\int_{-1}^{1}\frac{f(t)}{x+t}E(-v,t)\,\mathrm{d}t.$$

Now $E(v,t)$ is an even function of v, because by (24)

$$\begin{aligned}2E(-v,t) &= -(v+\mu)\mathrm{P}^{\mu}_{-v}(x)\mathrm{P}^{-\mu}_{-v-1}(-t) + (v-\mu)\mathrm{P}^{\mu}_{-v-1}(x)\mathrm{P}^{-\mu}_{-v}(-t)\\ &= -(v+\mu)\mathrm{P}^{\mu}_{v-1}(x)\mathrm{P}^{-\mu}_{v}(-t) + (v-\mu)\mathrm{P}^{\mu}_{v}(x)\mathrm{P}^{-\mu}_{v-1}(-t) = 2E(v,t).\end{aligned}$$

Therefore

$$\sum_{r=1}^{\infty} b_{-r}(v)\mathrm{P}^{\mu}_{v-r}(x) = \sum_{r=0}^{\infty} b_{-r-1}(v)\mathrm{P}^{\mu}_{v-r-1}(x) = \frac{1}{2\cos v\pi}\int_{-1}^{1}\frac{f(t)}{x+t}E(v,t)\,\mathrm{d}t,$$

as required.

Lemma 16. *If* $(1-t^2)^{-1/4}f(t) \in L(-1,1)$, *$f$ vanishes in a neighbourhood of a certain* $-x \in (-1,1)$, $|\operatorname{re}\mu| < \frac{1}{2}$, *$v$ is not half an odd integer, and b_n is defined as in* (1) (*and as in Theorem* 1 *below*), *then*

$$b_0\mathrm{P}^{\mu}_{v}(x) + \sum_{n=1}^{\infty}\{b_n\mathrm{P}^{\mu}_{v+n}(x) + b_{-n}\mathrm{P}^{\mu}_{v-n}(x)\} = 0.$$

Proof. The left-hand side is equal to

$$\sum_{n=0}^{\infty} b_n\mathrm{P}^{\mu}_{v+n}(x) + \sum_{n=1}^{\infty} b_{-n}\mathrm{P}^{\mu}_{v-n}(x),$$

because these two series are convergent, by Lemma 15. By that lemma also, their sum is zero, as required.

Remark. The next theorem is similar to [3, Theorem 3], but with the coefficients b_n replacing a_n and the bounded variation condition applied to a neighbourhood of $-x$ instead of one of x.

Theorem 1. *If* $(1-t^2)^{-1/4}f(t) \in L(-1,1)$, *$f$ has bounded variation on a neighbourhood of a certain* $-x \in (-1,1)$, $|\operatorname{re}\mu| < \frac{1}{2}$, *$v$ is not half an odd integer and*

$$b_n = \frac{v+n+\frac{1}{2}}{2\cos v\pi}\int_{-1}^{1} f(t)\mathrm{P}^{-\mu}_{v+n}(-t)\,\mathrm{d}t,$$

then

$$b_0\mathrm{P}^{\mu}_{v}(x) + \sum_{n=1}^{\infty} r^n\{b_n\mathrm{P}^{\mu}_{v+n}(x) + b_{-n}\mathrm{P}^{\mu}_{v-n}(x)\} \to 0 \quad \textit{as } r \to 1.$$

Proof. Let $\omega = \arccos x$; then $\cos(\pi-\omega) = -x$. In Lemma 12 let $a = \cos(\pi-\omega-2\varepsilon) = \cos\alpha$ and $b = \cos(\pi-\omega+2\varepsilon) = \cos\beta$, so that $-1 < b < -x < a < 1$, g has bounded variation on $[\alpha,\beta]$ and f has bounded variation on $[b,a]$. Let $f_k(t)$ $(k=1,2,3)$ be zero on $-1<t<1$ except that

$$f_1(t) - f(t) \quad \text{for } -1 < t \leqslant b, \qquad f_3(t) = f(t) \quad \text{for } a \leqslant t < 1$$

and

$$f_2(t) = f(t) \quad \text{for } b < t < a; \tag{26}$$

and let

$$b_n^{(k)} = \frac{v+n+\frac{1}{2}}{2\cos v\pi} \int_{-1}^{1} f_k(t) \mathrm{P}_{v+n}^{-\mu}(-t)\,\mathrm{d}t. \tag{27}$$

By Lemma 13,

$$b_0^{(2)} \mathrm{P}_v^{\mu}(x) + \sum_{n=1}^{\infty} r^n \{ b_n^{(2)} \mathrm{P}_{v+n}^{\mu}(x) + b_{-n}^{(2)} \mathrm{P}_{v-n}^{\mu}(x) \} \to 0 \quad \text{as } r \to 1. \tag{28}$$

Since $f_1(t)$ and $f_3(t)$ vanish on (b, a), Lemma 16 gives that

$$b_0^{(k)} \mathrm{P}_v^{\mu}(x) + \sum_{n=1}^{\infty} \{ b_n^{(k)} \mathrm{P}_{v+n}^{\mu}(x) + b_{-n}^{(k)} \mathrm{P}_{v-n}^{\mu}(x) \} = 0 \tag{29}$$

for $k = 1$ and 3. It follows from the consistency of Abel summation with convergence that, for these k,

$$b_0^{(k)} \mathrm{P}_v^{\mu}(x) + \sum_{n=1}^{\infty} r^n \{ b_n^{(k)} \mathrm{P}_{v+n}^{\mu}(x) + b_{-n}^{(k)} \mathrm{P}_{v-n}^{\mu}(x) \} \to 0 \quad \text{as } r \to 1. \tag{30}$$

Now $f_1(t) + f_2(t) + f_3(t) = f(t)$ for $-1 < t < 1$, and consequently $b_n^{(1)} + b_n^{(2)} + b_n^{(3)} = b_n$. Using these, addition of (28) and (30) (for $k = 1$ and 3) gives the stated conclusion of Theorem 1.

Lemma 17. *Under the hypotheses of Theorem* 1 *above, with* $f_2(t)$ *and* $b_n^{(2)}$ *defined as in* (26) *and* (27), *both*

$$b_{\pm n}^{(2)} \mathrm{P}_{v\pm n}^{\mu}(x) = \mathrm{O}(1/n) \quad \textit{as } n \to \infty.$$

Proof. Let $\alpha = \pi - \omega - 2\varepsilon = \arccos a$ and $\beta = \pi - \omega + 2\varepsilon = \arccos b$; also $\arccos(-x) = \pi - \omega \in (\alpha, \beta)$. By [1, 3.9(2)], applied with θ replaced by $\theta' = \pi - \theta$, μ by $-\mu$ and v by $v + n$,

$$\begin{aligned}
\int_{-1}^{1} f_2(t) \mathrm{P}_{v+n}^{-\mu}(-t)\,\mathrm{d}t &= \int_{\alpha}^{\beta} f_2(\cos\theta) \mathrm{P}_{v+n}^{-\mu}(\cos\theta') \sin\theta\,\mathrm{d}\theta \\
&= \frac{\Gamma(v+n-\mu+1)}{\Gamma(v+n+\frac{3}{2})} \sqrt{\frac{2}{\pi}} \int_{\alpha}^{\beta} f_2(\cos\theta) \sqrt{\sin\theta} \\
&\quad \times [\cos\{(v+n+\tfrac{1}{2})\theta' - (\mu+\tfrac{1}{2})\tfrac{1}{2}\pi\} + \mathrm{O}(n^{-1})]\,\mathrm{d}\theta
\end{aligned}$$

as $n \to \infty$. Let $l(\theta) = f_2(\cos\theta)\sqrt{\sin\theta}$; this is a function of bounded variation on $[\alpha, \beta]$ with $l(\alpha) = 0 = l(\beta)$. Integrating by parts,

$$\begin{aligned}
\int_{-1}^{1} f_2(t) \mathrm{P}_{v+n}^{-\mu}(-t)\,\mathrm{d}t &= \frac{\Gamma(v+n-\mu+1)}{\Gamma(v+n+\frac{3}{2})} \frac{\sqrt{2/\pi}}{v+n+\frac{1}{2}} \\
&\quad \times \left(\int_{\alpha}^{\beta} \sin\{(v+n+\tfrac{1}{2})\theta' - (\mu+\tfrac{1}{2})\tfrac{1}{2}\pi\}\,\mathrm{d}l(\theta) + \mathrm{O}(1) \right)
\end{aligned}$$

$$= \frac{\Gamma(v+n-\mu+1)}{\Gamma(v+n+\frac{3}{2})}\frac{1}{v+n+\frac{1}{2}}\,\mathrm{O}(1)$$

$$= \mathrm{O}(n^{-\mathrm{re}\,\mu-(3/2)})$$

as $n \to \infty$. By (27) with $k=2$, this makes $b_n^{(2)} = \mathrm{O}(n^{-\mathrm{re}\,\mu-(1/2)})$. By [2, Theorem 3],

$$b_n^{(2)}\mathrm{P}_{v+n}^{\mu}(x) = \mathrm{O}(n^{-\mathrm{re}\,\mu-(1/2)}n^{\mathrm{re}\,\mu-(1/2)}) = \mathrm{O}(n^{-1}).$$

The same estimate holds for $b_{-n}^{(2)}\mathrm{P}_{v-n}^{\mu}(x)$ because, by [1, 3.4(7)],

$$\mathrm{P}_{v-n}^{-\mu}(-t) = \mathrm{P}_{n-v-1}^{-\mu}(-t) \quad \text{and} \quad \mathrm{P}_{v-n}^{\mu}(x) = \mathrm{P}_{n-v-1}^{\mu}(x),$$

so that the above calculations apply with the constant v replaced by $-v-1$. This completes the proof of Lemma 17.

Remark. The next theorem is like [2, Theorem 10], the main difference being that the Dini condition is replaced by bounded variation on a neighbourhood; this permits f to have an ordinary discontinuity at $-x$.

Theorem 2. *If* $(1-t^2)^{-1/4}f(t) \in L(-1,1)$, f *has bounded variation on a neighbourhood of a certain* $-x \in (-1,1)$, $|\mathrm{re}\,\mu| < \frac{1}{2}$, v *is not half an odd integer and*

$$b_n = \frac{v+n+\frac{1}{2}}{2\cos v\pi}\int_{-1}^{1} f(t)\mathrm{P}_{v+n}^{-\mu}(-t)\,\mathrm{d}t,$$

then

$$b_0\mathrm{P}_v^{\mu}(x) + \sum_{n=1}^{\infty}\{b_n\mathrm{P}_{v+n}^{\mu}(x) + b_{-n}\mathrm{P}_{v-n}^{\mu}(x)\} = 0. \tag{31}$$

Proof. Lemma 17 permits Littlewood's Tauberian Theorem [4] to be applied to (28), giving that

$$b_0^{(2)}\mathrm{P}_v^{\mu}(x) + \sum_{n=1}^{\infty}\{b_n^{(2)}\mathrm{P}_{v+n}^{\mu}(x) + b_{-n}^{(2)}\mathrm{P}_{v-n}^{\mu}(x)\} = 0.$$

Adding to this the two equations (29), which hold for $k=1$ and 3, gives the conclusion of Theorem 2.

Remark. Theorem 2 is the final (in this paper) version of the "expansion of zero" mentioned in the Introduction.

Theorem 3. *If* $(1-t^2)^{-1/4}f(t) \in L(-1,1)$, $x \in (-1,1)$, f *has bounded variation on neighbourhoods of both* x *and* $-x$, $|\mathrm{re}\,\mu| < \frac{1}{2}$, v *is not half an odd integer and*

$$c_n = \frac{v+n+\frac{1}{2}}{\cos v\pi}\int_{-1}^{1} f(t)\mathrm{P}_{v+n}^{-\mu}(-t)\,\mathrm{d}t,$$

then

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = c_0\mathrm{P}_v^{\mu}(x) + \sum_{n=1}^{\infty}\{c_{2n}\mathrm{P}_{v+2n}^{\mu}(x) + c_{-2n}\mathrm{P}_{v-2n}^{\mu}(x)\}.$$

Proof. By [3, Theorem 4],

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}=a_0\mathrm{P}_\nu^\mu(x)+\sum_{n=1}^{\infty}\{a_n\mathrm{P}_{\nu+n}^\mu(x)+a_{-n}\mathrm{P}_{\nu-n}^\mu(x)\}, \tag{32}$$

where, as in [3, Theorem 2], $a_n=(-1)^n b_n$. Adding to (32) the conclusion (31) of Theorem 2,

$$\begin{aligned}\tfrac{1}{2}\{f(x+0)+f(x-0)\}&=(a_0+b_0)\mathrm{P}_\nu^\mu(x)+\sum_{n=1}^{\infty}\{(a_n+b_n)\mathrm{P}_{\nu+n}^\mu(x)+(a_{-n}+b_{-n})\mathrm{P}_{\nu-n}^\mu(x)\}\\ &=2b_0\mathrm{P}_\nu^\mu(x)+\sum_{n=1}^{\infty}\{((-1)^n+1)b_n\mathrm{P}_{\nu+n}^\mu(x)+((-1)^{-n}+1)b_{-n}\mathrm{P}_{\nu-n}^\mu(x)\}\\ &=2b_0\mathrm{P}_\nu^\mu(x)+\sum_{n=1}^{\infty}\{2b_{2n}\mathrm{P}_{\nu+2n}^\mu(x)+2b_{-2n}\mathrm{P}_{\nu-2n}^\mu(x)\};\end{aligned}$$

this is equivalent to the stated conclusion, since $2b_n=c_n$.

Appendix

Clearly, any linear combination of (31) and (32) will give an expansion theorem under the hypotheses of Theorem 3; but the only ones which will omit every second Legendre function are those obtained by adding and subtracting (31) and (32). Theorem 3 is the result of adding them. The result of subtracting them is, as in the proof of Theorem 3,

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}=(a_0-b_0)\mathrm{P}_\nu^\mu(x)+\sum_{n=1}^{\infty}\{(a_n-b_n)\mathrm{P}_{\nu+n}^\mu(x)+(a_{-n}-b_{-n})\mathrm{P}_{\nu-n}^\mu(x)\}.$$

Now

$$a_n-b_n=((-1)^n-1)b_n=\begin{cases}0 & \text{if } n \text{ is even},\\ -2b_n=-c_n & \text{if } n \text{ is odd}.\end{cases}$$

It is easily verified that

$$-c_{2n-1}=\frac{(\nu-1)+2n+\frac{1}{2}}{\cos(\nu-1)\pi}\int_{-1}^{1}f(t)\mathrm{P}_{(\nu-1)+2n}^{-\mu}(-t)\,\mathrm{d}t\ =c_{+2n}^{-}\quad\text{say}$$

and

$$-c_{-2n+1}=\frac{(\nu+1)-2n+\frac{1}{2}}{\cos(\nu+1)\pi}\int_{-1}^{1}f(t)\mathrm{P}_{(\nu+1)-2n}^{-\mu}(-t)\,\mathrm{d}t\ =c_{-2n}^{+}\quad\text{say.}$$

Evidently

$$\begin{aligned}&c_{+2n}^{-}\text{ is the result of replacing } \nu \text{ by } \nu-1 \text{ in } c_{+2n},\\ &c_{-2n}^{+}\text{ is the result of replacing } \nu \text{ by } \nu+1 \text{ in } c_{-2n}.\end{aligned} \tag{33}$$

The expansion obtained is therefore

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}=\sum_{n=1}^{\infty}\{c^{-}_{+2n}\mathrm{P}^{\mu}_{v+2n-1}(x)+c^{+}_{-2n}\mathrm{P}^{\mu}_{v-2n+1}(x)\}. \tag{34}$$

This is clearly different from the expansion in Theorem 3, in that it contains exactly those P^{μ}_{v+n} which are omitted in that theorem. I give it less prominence because it has less simplicity and elegance than Theorem 3.

On the other hand, replacing v by $v+1$ in (34) gives the series

$$\sum_{n=1}^{\infty}\{c_{2n}\mathrm{P}^{\mu}_{v+2n}(x)+c^{++}_{-2n}\mathrm{P}^{\mu}_{v+2-2n}(x)\}, \tag{35}$$

where c^{++}_{-2n} is the result of replacing v by $v+2$ in c_{-2n}, that is,

$$c^{++}_{-2n}=\frac{v-2(n-1)+\frac{1}{2}}{\cos v\pi}\int_{-1}^{1}f(t)\mathrm{P}^{-\mu}_{v-2(n-1)}(-t)\,\mathrm{d}t,\ =c_{-2(n-1)}.$$

Thus (35) is

$$\sum_{n=1}^{\infty}\{c_{2n}\mathrm{P}^{\mu}_{v+2n}(x)+c_{-2(n-1)}\mathrm{P}^{\mu}_{v-2(n-1)}(x)\}. \tag{36}$$

If (and only if) $\sum_{n=1}^{\infty}c_{2n}\mathrm{P}_{v+2n}(x)$ is convergent, (36) can be separated into the sum of two series, becoming equal to

$$\begin{aligned}\sum_{n=1}^{\infty}c_{2n}\mathrm{P}^{\mu}_{v+2n}(x)+\sum_{n=1}^{\infty}c_{-2(n-1)}\mathrm{P}^{\mu}_{v-2(n-1)}(x)&=\sum_{n=1}^{\infty}c_{2n}\mathrm{P}^{\mu}_{v+2n}(x)+\sum_{n=0}^{\infty}c_{-2n}\mathrm{P}^{\mu}_{v-2n}(x)\\&=c_0\mathrm{P}^{\mu}_{v}(x)+\sum_{n=1}^{\infty}\{c_{2n}\mathrm{P}^{\mu}_{v+2n}(x)+c_{-2n}\mathrm{P}^{\mu}_{v-2n}(x)\},\end{aligned}$$

which is just the expansion in Theorem 3.

Thus, the expansion (34) is equivalent to that in Theorem 3, by replacing v by $v+1$, certainly whenever the series

$$\sum_{n=1}^{\infty}c_{2n}\mathrm{P}_{v+2n}(x)$$

is convergent. By [2, Theorem 11] this is indeed the case when f is Dini at both x and $-x$.

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